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IN SEARCH OF THE SHAPE OF THE UNIVERSE

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Motivation

So what is the shape of the universe? We know the shape of the Earth - we can (in theory) go out into space, look back on Earth and see it is nearly a sphere. This *extrinsic* way of classifying shapes is intuitive to us, because we use it all the time...but we can't do this when we are thinking about the universe, because we can't go 'outside' it. It could be flat or spherical...or maybe a doughnut shape? Or a pear shape? Maybe it's like a Klein bottle, or something even more exotic? After all, if we were stuck on the surface of these shapes, and these shapes were big enough, we would not be able to tell the difference because all we would see is 'flat' land.

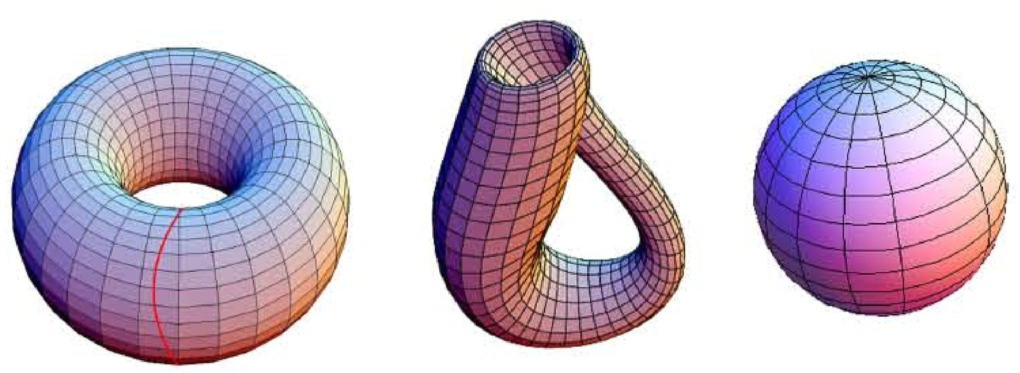

Fig. 1: A *2-torus*, a *Klein bottle*, and a *2-sphere*. Note these are 'hollow'; interior points are not part of the objects. When we are on Earth we view our land as being flat, even though we know the Earth is nearly spherical.

We will see how a study of geometry provides us with some tools to help us answer this seemingly untouchable question.

Some geometric ideas

To extrinsically study shapes, we first put the shape into some Euclidean (i.e. flat) space, and then we study its geometry. For example, we put a circle into the 2-dimensional (x,y) plane, and from that we can find out its circumference and area. The circle is a 1-dimensional object: it is completely described by how far along the curve we are from some reference point we provide. By a similar argument, the sphere in Fig. 1 is a 2-dimensional object sitting in a 3-dimensional Euclidean space, because where we are on the sphere is completely described by specifying the longitude and the latitude.

We can generalise this: an *n-manifold* is an n-dimensional space which is locally 'Euclidean', or 'flat'. Curves are 1-manifolds, surfaces are 2-manifolds (see Fig. 1), and we can extend this to higher dimensions. For example, we can easily construct the n-sphere which is the n-dimensional analogue of the 2-sphere; although we can't easily visualise it, it is by construction an n-manifold. A manifold is *closed* if it has no boundary: for example, the 2-sphere is closed whereas a disk (see Fig. 2) is not because it has a circle as its boundary. An n-manifold may be embedded into an m-Euclidean space (m greater than n), where we can study its geometry extrinsically.

The reason for introducing this notion is that usually we view our universe as a (3+1)-space consisting of three spatial dimensions and one time dimension. The idea therefore is to model the spatial portion of the universe as a 3-manifold. Once we have a model, we compare the physical data obtained through observation against the theoretical model; if a particular model is not consistent with the data, then it is very likely that this model does not represent the spatial portion of the universe.

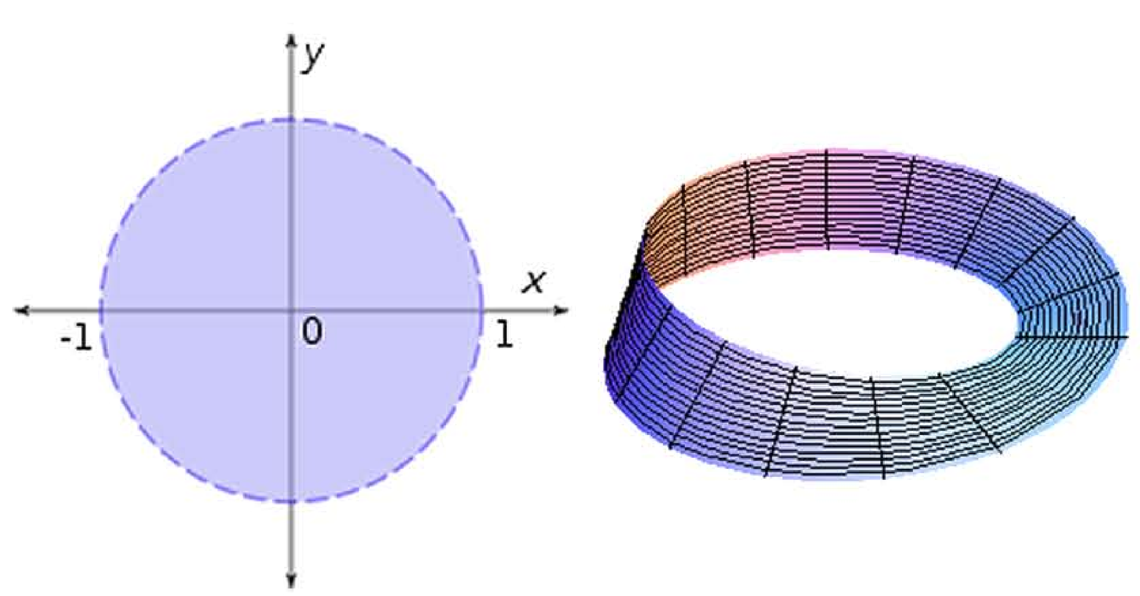

Fig. 2: Some 2-manifolds with boundaries: the disk and the Möbius strip. The 2-manifolds in Fig. 1 by comparison are closed manifolds without boundaries.

We have used associated the word 'flat' with the Euclidean space. However, a lot of things are not flat. We know that on a flat space, the shortest way to travel between two points is via a straight line, but what happens when the manifold we consider is not flat? This is where notions of *curvature* and *geodesics* are important.

Curvature arises from the bending of manifolds, and we use it as a measure of how much a manifold deviates from our flat Euclidean space which has curvature zero. For example, the unit 2-sphere has curvature one (it bends 'positively'). Lines in Euclidean space become curves on a curved manifold. *Geodesics* are then the corresponding objects to straight lines in Euclidean space: the curve of minimum distance joining two points on a manifold is a geodesic. We can visualise this by drawing a straight line on a piece of paper and rolling it up into a cylinder. This now gives a curve which is still distance minimising; depending on how you drew the line, you will either get a straight line, part of a circle, or part of a *helix*, and these are all geodesics on the cylinder.

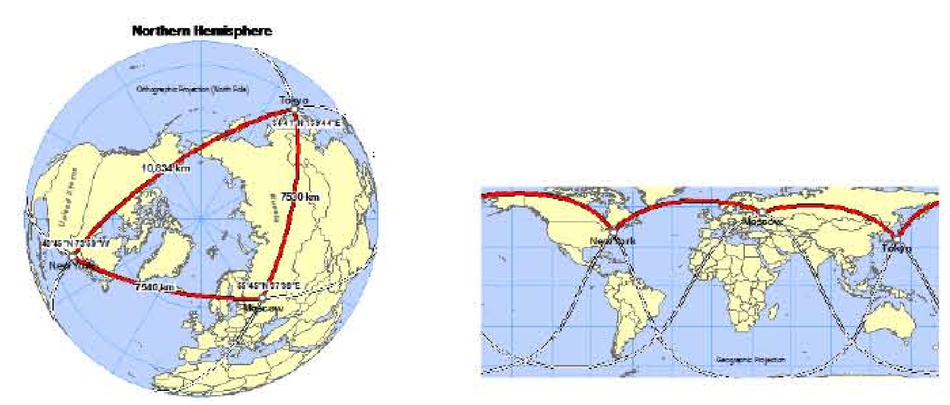

Fig. 3: Geodesics on a sphere are portions of circles where the centre of the circle and the centre of the sphere coincide; these are called *great circles*. This diagram shows flight paths used in air travel and geodesics here are important because they reduce the cost of air travel. (Image courtesy of J.P. Rodrigue, Hofstra University)

The idea of Riemann

So far geometry has been investigated from an outsider's point of view, in the sense that we take the manifold we are interested in, consider an embedding into Euclidean space and study its geometry. Could we do geometry from an *intrinsic* point of view rather than the *extrinsic* point of view?

Fig. 4: Bernhard Riemann (1826 - 1866)

The mathematician Carl Friedrich Gauss certainly thought intrinsic geometry was possible; it was he who showed that curvature was an intrinsic property of a surface (i.e. a 2-manifold), independent of the embedding. However, it was Gauss' student Bernhard Riemann who first gave a formal treatment of studying geometry intrinsically. Riemann showed geometry is still consistent if we only have the manifold in question and a way to measure distances on this manifold (the *metric*); how it is embedded in Euclidean space does not affect its geometry. Curvatures, geodesics and other geometric objects can then be recast into the Riemannian formulation, and geometry may be studied from within. This extra freedom makes the formulation more complicated to set up, but it is essential for us; we can't go outside the universe, so there is no way we could study the universe extrinsically. Indeed, this is the approach Einstein adopted when he formulated his theory of *general relativity*.

Other consistent geometries which behave quite differently to our standard notion of Euclidean and spherical geometry have been constructed since. One area of current research is that of *hyperbolic* geometry which studies spaces with negative curvature; one example of such space is the *pseudosphere* in Fig. 7. Sometimes called *horn* geometries, it is one of the proposed models of the universe and data obtained currently does not rule out the possibility of a hyperbolic universe.

Local vs. Global

Geometry is concerned with studying things *locally*. For example, when studying the 2-torus in Fig. 1, we can find geodesics and its curvature, but this does not tell us anything about the fact that it has a 'hole' in it; it is not *simply connected* like the 2-sphere is, because the 2-sphere has no 'holes'. *Topology* studies global properties of the space. Topological invariants are obtained as a result of this study, and these provide powerful classification theorems which allows us to make deductions. Topology is sometimes called rubber-sheet geometry: two manifolds are the 'same' if each can be smoothly deformed into the other. We call two manifolds *homeomorphic* if there is a smooth deformation (the *homeomorphism*) which preserves all the topological invariants as we deform one manifold into the other.

Fig. 5: A coffee mug and a doughnut: these two objects are homeomorphic. (Image by L.V. Barbosa)

For example, we can smoothly deform the shapes in Fig. 5 from one into one another without tearing the shape, and these two objects can be shown to be homeomorphic. In Fig. 1, the 2-torus and the 2-sphere are hot homeomorphic, since there is no way we can deform one into another without tearing the surface. Also, the *genus* (the number of 'holes') is not preserved if we forcibly deform our 2-torus into the 2-sphere. The genus is one of the topological invariants, and so we conclude the 2-torus and the 2-sphere are fundamentally different. There are many classification theorems arising from the study of topology, but perhaps the most famous result of this century is the *Poincaré conjecture*.

The million dollar question

"Consider a connected 3-manifold without boundary and finite in size. If every simple closed loop on this manifold can be continuously shrunk down to a point, then this manifold is homeomorphic to the 3-sphere."

This is the *Poincaré conjecture* for a 3-manifold. We will briefly explain this for the 2-manifold analogue:

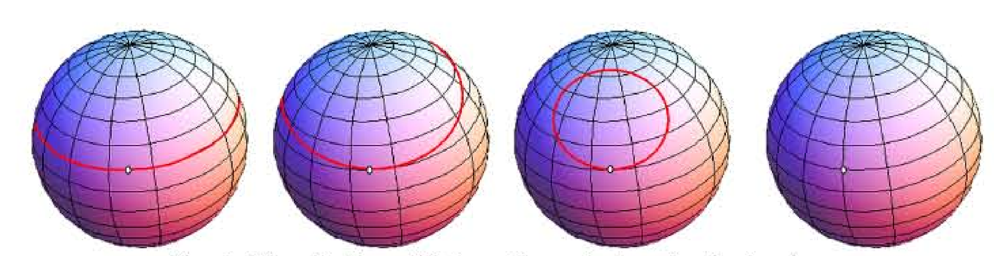

Fig. 6: Visualisation of Poincaré's conjecture for the 2-sphere.

The 2-sphere has no holes, is finite in size and lacks a boundary, hence it satisfies our imposed conditions. Every simply connected loop (loop with no self-intersections) can be shrunk down to a point, and the 2-sphere is clearly homeomorphic to itself. This is not true for the 2-torus, because if we take the red loop as shown in Fig. 1, then this loop cannot be shrunk down to a point.

This very intuitive classification theorem was first proposed by Henri Poincaré around the turn of the twentieth century, and was selected by the Clay Institute of Mathematics as one of the seven *Millennium Prize Questions*, with one million US dollars attached to each prize. The 3-dimensional case was solved by Grigori Perelman and confirmed as a proof only in 2006. It remains the only solved Millennium problem.

Apart from classification theorems, there are also local to global theorems which gives us global results knowing something about the local geometry. This also reduces some possibilities of the manifolds we need to consider. One such is the *Gauss-Bonnet* theorem, which relates curvature to the genus of the surface. The upshot is that we have powerful tools from mathematics which help us study the geometry and topology of the universe.

Universe models and their consequences

Current models of the universe are based around simply connected and closed 3-manifolds which are spherical (positive curvature), Euclidean (zero curvature) or hyperbolic (negative curvature). By comparing physical data obtained through experiments and observations against the classification theorems, we can see whether the data agree with the theories and we can eliminate inconsistent models as appropriate. This allows us to restrict the cases we need to consider.

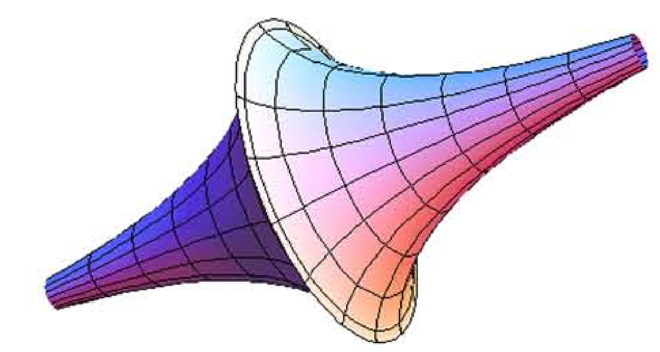

Fig. 7: The unit *pseudosphere*. With the associated metric in hyperbolic space, this object has curvature -1 away from its equator.

The shape of the universe has important implications: it is one of the important factors that determines the ultimate fate of the universe. Scientists believe that the universe is currently expanding; observation of *redshifts* in electromagnetic radiation supports this. Theoretical studies suggest that if the universe is spherical and there is enough mass in the universe, then gravity will eventually reverse this expansion and the *big crunch* will result (the universe shrinking down to a point). On the other hand, if the universe is flat or hyperbolic, then this will not happen and the universe will continue to expand, leading to either the *big freeze* (the average energy density dropping to a point where life cannot be sustained) or the *big rip* (the repulsive forces pushing the universe apart overcome the forces binding molecules together and everything rips apart). The geometry of the universe is clearly an important area that requires more investigation!

Conclusion

Experimental results show that the universe is very nearly flat, and if there is curvature, it is certainly very small. The two models currently favoured by researchers are the *Poincaré dodecahedral space* for the spherical case (generally known as soccer ball shaped) and the *Picard horn* for the hyperbolic case (generally known as funnel shaped); the universe is probably not flat. We have assumed here that the universe is a simply connected 3-manifold without boundary in a (3+1)-space; this need not be the case. There are other studies which investigate models with different dimensions or topology and these give rise to some very curious physical consequences. The shape of the universe is still very much an open question.

The ancients believed the world was balanced on top of four elephants, themselves balanced on top of a turtle. Although we can't provide an answer to our initial question, at least we can say the universe is extremely unlikely to be balanced on top of a gigantic turtle. Interesting idea though.

Fig. 8: Turtle universe?

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